

# A PHASE TRANSITION IN A CONTINUUM CURIE-WEISS SYSTEM WITH BINARY INTERACTIONS

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**ABSTRACT.** A single-sort continuum Curie-Weiss system of interacting particles is studied. The particles are placed in the space  $\mathbb{R}^d$  divided into congruent cubic cells. For a region  $V \subset \mathbb{R}^d$  consisting of  $N \in \mathbb{N}$  cells, each two particles contained in  $V$  attract each other with intensity  $J_1/N$ . The particles contained in the same cell are subject to binary repulsion with intensity  $J_2 > J_1$ . For fixed values of the temperature, the interaction intensities and the chemical potential, the thermodynamic phase is defined as a probability measure on the space of occupation numbers of cells, determined by a condition, typical to Curie-Weiss theories. There is shown that the half-plane  $J_1 \times \text{chemical potential}$  contains phase coexistence points, and thus multiple thermodynamic phases of the system may exist. An equation of state for this system is obtained.

## 1. INTRODUCTION

The mathematical theory of phase transitions in continuum particle systems has much fewer results as compared to its counterpart dealing with discrete underlying sets like lattices, graphs, etc. It is quite natural that the first steps in such theories are made by employing various mean field models. In [1], the mean field approach was mathematically realized by using a Kac-like infinite range attraction combined with a two-body repulsion. By means of rigorous upper and lower bounds for the canonical partition function obtained in that paper the authors derived the equation of state indicating the possibility of a first-order phase transition. Later on, this result was employed in [2], see also [3] and [4, 5] for recent results, to go beyond the mean field. Another way of realizing the mean-field approach is to use Curie-Weiss interactions and then appropriate methods of calculating asymptotics of integrals. Quite recently, this way was formulated as a coherent mathematical theory based on the large deviation techniques, in the framework of which the Gibbs states (thermodynamic phases) of the system are constructed as probability measures on an appropriate phase space, see [6, Section II].

In this work, we introduce a simple Curie-Weiss type model of a single-sort continuum particle system in which the space  $\mathbb{R}^d$  is divided into congruent (cubic) cells. For a bounded region  $V \subset \mathbb{R}^d$  consisting of  $N$  such cells, the attraction between each two particles in  $V$  is set to be  $J_1/N$ , regardless their positions. If such two particles lie in the same cell, they repel each other with intensity  $J_2 > J_1$ . Unlike to [1], we are going to deal with the grand canonical ensemble. Therefore, our initial thermodynamic variables are the inverse temperature  $\beta = 1/k_B T$  and the physical chemical potential. However, for the sake of convenience we employ the variables  $p = \beta J_1$  and  $\mu = \beta \times \text{physical chemical potential}$  and define single-phase domains of the half-plane  $\{(p, \mu) : p > 0, \mu \in \mathbb{R}\}$ , see Definition 2.3, by a condition that guaranties the existence of a unique  $\bar{y} \in \mathbb{R}$ , which determines a probability measure  $\mathbf{Q}_{p,\mu}$ , given in (2.22) and (2.21). In the grand canonical formalism and the approach of [6], this measure is set to be the thermodynamic phase of the system. The points  $(p, \mu)$  where the mentioned single-phase condition fails to hold due to the existence of multiple  $\bar{y}$  correspond to the coexistence of multiple thermodynamic phases. In Theorem 2.9, we show that there exists  $p_0 > 0$  such that  $\mathcal{R}(p_0) := \{(p, \mu) : p \in (0, p_0), \mu \in \mathbb{R}\}$  is

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a single-phase domain, that is, there is no phase-coexistence points in the strip  $\mathcal{R}(p_0)$  for sufficiently small attractions. Note that, for some models on graphs, see, e.g., the example given in [7], there exist multiple phases for each positive attraction. Thereafter, in Theorem 2.10, we show that, for each value of  $J_2/J_1 := a > 1$ , there exist points in which two phases do coexist. Namely, we show that there exists  $p_1 > 0$  such that, for each  $p \geq p_1$ , there exists  $\mu_c(p) \in \mathbb{R}$  and small enough  $\epsilon > 0$  such that the sets  $\{(p, \mu) : \mu \in (\mu_c(p) - \epsilon, \mu_c(p))\}$  and  $\{(p, \mu) : \mu \in (\mu_c(p), \mu_c(p) + \epsilon)\}$  lie in different single-phase domains, and that there exist at least two  $\bar{y}$  whenever  $\mu = \mu_c(p)$ . In Section 3, we present a number of numerical calculations which illustrate the facts proved in Theorems 2.9 and 2.10.

## 2. THE MODEL

By  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  we denote the sets of natural, real and complex numbers, respectively. We also put  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For  $d \in \mathbb{N}$ , by  $\mathbb{R}^d$  we denote the Euclidean space of vectors  $x = (x^1, \dots, x^d)$ ,  $x^i \in \mathbb{R}$ . In the sequel, its dimension  $d$  will be fixed. By  $dx$  we mean the Lebesgue measure on  $\mathbb{R}^d$ . In this section, we use some tools of the analysis on configuration spaces the main aspects of which can be found in [8].

**2.1. The integration.** For  $n \in \mathbb{N}$ , let  $\Gamma^{(n)}$  be the set of all  $n$ -point subsets of  $\mathbb{R}^d$ . Each such a subset,  $\gamma \in \Gamma^{(n)}$ , is an  $n$ -element set of distinct vectors  $x \in \mathbb{R}^d$ . We will call  $\gamma$  a *configuration*. Let also  $\Gamma^{(0)}$  be the one-element set consisting of the empty configuration. Each  $\Gamma^{(n)}$  is equipped with the topology related to the Euclidean topology of  $\mathbb{R}^d$ . Then we define

$$\Gamma_0 = \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)},$$

that is,  $\Gamma_0$  is the topological sum of the spaces  $\Gamma^{(n)}$ . We equip  $\Gamma_0$  with the corresponding Borel  $\sigma$ -field  $\mathcal{B}(\Gamma_0)$  which makes  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  a standard Borel space. A function  $G : \Gamma_0 \rightarrow \mathbb{R}$  is  $\mathcal{B}(\Gamma_0)$ -measurable if, for each  $n \in \mathbb{N}$ , there exists a symmetric Borel function  $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$  such that  $G(\gamma) = G^{(n)}(x_1, \dots, x_n)$  for  $\gamma = \{x_1, \dots, x_n\}$ . For such a function, we also set  $G^{(0)} = G(\emptyset)$ . The Lebesgue-Poisson measure  $\lambda$  on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  is defined by the relation

$$\int_{\Gamma_0} G(\gamma) \lambda(d\gamma) = G^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \quad (2.1)$$

which has to hold for all measurable  $G : \Gamma_0 \rightarrow \mathbb{R}_+ := [0, +\infty)$ .

For some  $c > 0$ , we let  $\Delta = (-c/2, c/2]^d \subset \mathbb{R}^d$  be a cubic cell of volume  $v = c^d$  centered at the origin. Let also  $V \subset \mathbb{R}^d$  be the union of  $N \in \mathbb{N}$  disjoint translates  $\Delta_\ell$  of  $\Delta$ , i.e.,

$$V = \bigcup_{\ell=1}^N \Delta_\ell.$$

As is usual for Curie-Weiss theories, cf. [6], the form of the interaction energy of the system of particles placed in  $V$  depends on  $V$ . In our model, the energy of a configuration  $\gamma \subset V$  is

$$\begin{aligned} W_N(\gamma) &= \frac{1}{2} \sum_{x, y \in \gamma} \Phi_N(x, y), \\ \Phi_N(x, y) &= -J_1/N + J_2 \sum_{\ell=1}^N \mathbb{I}_{\Delta_\ell}(x) \mathbb{I}_{\Delta_\ell}(y), \end{aligned} \quad (2.2)$$

where  $\mathbb{I}_{\Delta_\ell}$  is the indicator of  $\Delta_\ell$ , that is,  $\mathbb{I}_{\Delta_\ell}(x) = 1$  if  $x \in \Delta_\ell$  and  $\mathbb{I}_{\Delta_\ell}(x) = 0$  otherwise. For convenience, in  $W_N$  above we have included the self-interaction term  $\Phi_N(x, x)$ , which does not affect the physics of the model. We also write  $W_N$  and  $\Phi_N$  instead of writing

$W_V$  and  $\Phi_V$  since these quantities depend only on the number of cells in  $V$  but not on its particular location. The first term in  $\Phi_N$  with  $J_1 > 0$  describes attraction. By virtue of the Curie-Weiss approach, it is taken equal for all particles. The second term with  $J_2 > 0$  describes repulsion between two particles contained in one and the same cell. That is, in our model every two particles in  $V$  attract each other independently of their location, and repel if they are in the same cell. The intensities  $J_1$  and  $J_2$  are assumed to satisfy the following condition

$$J_2 > J_1. \quad (2.3)$$

The latter is to secure the stability of the interaction, see [9], that is to satisfy

$$\int_V \Phi_N(x, y) dy > 0, \quad \text{for all } x \in V.$$

Let  $\beta = 1/k_B T$  be the inverse temperature. To optimize the thermodynamic variables we introduce the following

$$p = \beta J_1, \quad a = J_2/J_1, \quad (2.4)$$

and the dimensionless chemical potential  $\mu = \beta \times (\text{physical chemical potential})$ . Then  $(p, \mu) \in \mathbb{R}_+ \times \mathbb{R}$  is considered as the basic set of thermodynamic variables, whereas  $a$  and  $v$  are model parameters.

The grand canonical partition function in region  $V$  is

$$\begin{aligned} \Xi_N(p, \mu) &= \int_{\Gamma_V} \exp \left( \mu |\gamma| - \beta W_N(\gamma) \right) \lambda(d\gamma), \\ &= \int_{\Gamma_V} \exp \left( \mu |\gamma| + \frac{p}{2N} |\gamma|^2 - \frac{ap}{2} \sum_{x, y \in \gamma} \sum_{\ell=1}^N \mathbb{I}_{\Delta_\ell}(x) \mathbb{I}_{\Delta_\ell}(y) \right) \lambda(d\gamma), \end{aligned} \quad (2.5)$$

where  $|\gamma|$  stands for the number of points in configuration  $\gamma$  and  $\Gamma_V$  is the subset of  $\Gamma_0$  consisting of all  $\gamma$  contained in  $V$ . We write  $\Xi_N$  instead of  $\Xi_V$  for the reasons mentioned above.

**2.2. Transforming the partition function.** Now we use the concrete form of the energy as in (2.2) to bring (2.5) to a more convenient form. For a given  $\ell = 1, \dots, N$  and a configuration  $\gamma \in \Gamma_V$ , we set  $\gamma_\ell = \gamma \cap \Delta_\ell$ , that is,  $\gamma_\ell$  is the part of the configuration contained in  $\Delta_\ell$ . Then  $|\gamma_\ell|$  will stand for the number of points of  $\gamma$  contained in  $\Delta_\ell$ . Note that

$$|\gamma_\ell| = \sum_{x \in \gamma_\ell} 1 = \sum_{x \in \gamma} \mathbb{I}_{\Delta_\ell}(x). \quad (2.6)$$

Then, cf. (2.2) and (2.5),

$$\begin{aligned} \sum_{x, y \in \gamma} \Phi_N(x, y) &= \sum_{\ell, \ell'=1}^N \sum_{x \in \Delta_\ell} \sum_{y \in \Delta_{\ell'}} \Phi_N(x, y) \\ &= -\frac{J_1}{N} \left( \sum_{\ell=1}^N |\gamma_\ell| \right)^2 + J_2 \sum_{\ell=1}^N |\gamma_\ell|^2 \end{aligned}$$

Thereby, the integrand in (2.5) can be rewritten in the following form. Set

$$F_N(\varrho, p, \mu) = \exp \left( \frac{p}{2N} \left( \sum_{\ell=1}^N \varrho_\ell \right)^2 + \mu \sum_{\ell=1}^N \varrho_\ell - \frac{ap}{2} \sum_{\ell=1}^N \varrho_\ell^2 \right), \quad (2.7)$$

where  $\varrho \in \mathbb{N}_0^N$  is a vector with nonnegative integer components  $\varrho_\ell$ ,  $\ell = 1, 2, \dots, N$ . Then the integral in (2.5) takes the form

$$\Xi_N(p, \mu) = \int_{\Gamma_V} F_N(\nu(\gamma), p, \mu) \lambda(d\gamma), \quad (2.8)$$

where  $F_N$  is as in (2.7) and  $\nu(\gamma) \in \mathbb{N}_0^N$  is the vector with component  $|\gamma_\ell|$ ,  $\ell = 1, \dots, N$ . For  $n, m \in \mathbb{N}_0$ , the Kronecker  $\delta$ -symbol can be written

$$\delta_{nm} = \int_0^1 \exp\left(2\pi i t(n - m)\right) dt, \quad i = \sqrt{-1}.$$

Applying this in (2.8) we get

$$\begin{aligned} \Xi_N(p, \mu) &= \sum_{\varrho \in \mathbb{N}_0^N} F_N(\varrho, p, \mu) \\ &\times \int_{\Gamma_V} \int_{[0,1]^N} \exp\left(2\pi i \sum_{\ell=1}^N (\varrho_\ell - |\gamma_\ell|) t_\ell\right) \lambda(d\gamma) dt_1 \cdots dt_N \\ &= \sum_{\varrho \in \mathbb{N}_0^N} F_N(\varrho, p, \mu) \\ &\times \int_{[0,1]^N} \exp\left(2\pi i \sum_{\ell=1}^N \varrho_\ell t_\ell\right) R_N(t_1, \dots, t_N) dt_1 \cdots dt_N. \end{aligned} \quad (2.9)$$

Here

$$\begin{aligned} R_N(t_1, \dots, t_N) &= \int_{\Gamma_V} \exp\left(-2\pi i \sum_{\ell=1}^N |\gamma_\ell| t_\ell\right) \lambda(d\gamma) \\ &= \int_{\Gamma_V} \exp\left(-2\pi i \sum_{\ell=1}^N \sum_{x \in \gamma} \mathbb{I}_{\Delta_\ell}(x) t_\ell\right) \lambda(d\gamma) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{V^n} \exp\left(-2\pi i \sum_{\ell=1}^N \sum_{j=1}^n \mathbb{I}_{\Delta_\ell}(x_j) t_\ell\right) dx_1 \cdots dx_n. \end{aligned} \quad (2.10)$$

In getting the second line of (2.10), we use (2.6), and then the integral with  $\lambda$  is written according to (2.1). Note that the expression under the integral in the last line of (2.10) factors in  $j$ , which allows for writing it in the form

$$\begin{aligned} R_N(t_1, \dots, t_N) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_V \exp\left(-2\pi i \sum_{\ell=1}^N \mathbb{I}_{\Delta_\ell}(x) t_\ell\right) dx \right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{\ell=1}^N \int_{\Delta_\ell} \exp\left(-2\pi i t_\ell\right) dx \right)^n \\ &= \exp\left(v \sum_{\ell=1}^N \exp\left(-2\pi i t_\ell\right)\right). \end{aligned}$$

Now we apply this in (2.9) and obtain

$$\begin{aligned}\Xi_N(p, \mu) &= \sum_{\varrho \in \mathbb{N}_0^N} F_N(\varrho, p, \mu) \prod_{\ell=1}^N \left( \frac{v^{\varrho_\ell}}{\varrho_\ell!} \right) \\ &= \sum_{\varrho \in \mathbb{N}_0^N} \exp \left( \frac{p}{2N} \left( \sum_{\ell=1}^N \varrho_\ell \right)^2 \right) \prod_{\ell=1}^N \pi(\varrho_\ell, \mu),\end{aligned}\quad (2.11)$$

where  $p$  is as in (2.4) and

$$\pi(n, \mu) = \frac{v^n}{n!} \exp \left( \mu n - \frac{1}{2} a p n^2 \right), \quad n \in \mathbb{N}_0. \quad (2.12)$$

Note that, for  $p = 0$ ,  $\pi$  turns into the (non-normalized) Poisson distribution with parameter  $ve^\mu$ . Hence, alternating the cell size amounts to shifting  $\mu$ .

**2.3. Single-phase domains.** By a standard identity involving Gaussian integrals we have

$$\exp \left( \frac{p}{2N} \left( \sum_{\ell=1}^N \varrho_\ell \right)^2 \right) = \sqrt{\frac{N}{2\pi p}} \int_{\mathbb{R}} \exp \left( -N \frac{y^2}{2p} + y \sum_{\ell=1}^N \varrho_\ell \right) dy.$$

Applying this in (2.11) we arrive at

$$\Xi_N(p, \mu) = \sqrt{\frac{N}{2\pi p}} \int_{\mathbb{R}} \exp \left( N E(y, p, \mu) \right) dy, \quad (2.13)$$

where

$$E(y, p, \mu) = -\frac{y^2}{2p} + \ln K(y, p, \mu), \quad (2.14)$$

and, cf. (2.4) and (2.12),

$$K(y, p, \mu) = \sum_{n=0}^{\infty} \frac{v^n}{n!} \exp \left( (y + \mu)n - \frac{ap}{2} n^2 \right). \quad (2.15)$$

Note that  $E$  is an infinitely differentiable function of all its arguments. Set

$$P_N(p, \mu) = \frac{1}{vN} \ln \Xi_N(p, \mu). \quad (2.16)$$

By the following evident inequality

$$(y + \mu)n - \frac{ap}{2} n^2 \leq \frac{(y + \mu)^2}{2ap}, \quad n \in \mathbb{N}_0,$$

we obtain from (2.15) and (2.14) that

$$E(y, p, \mu) \leq -\frac{a-1}{2ap} y^2 + \frac{\mu}{2ap} (2y + \mu) + v. \quad (2.17)$$

By virtue of Laplace's method [10], to calculate the large  $N$  limit in (2.16) we have to find global maxima of  $E(y, p, \mu)$  as a function of  $y \in \mathbb{R}$ .

*Remark 2.1.* From the estimate in (2.17) it follows that: (a) the integral in (2.13) is convergent for all  $p > 0$  and  $\mu \in \mathbb{R}$  since  $a > 1$ , see (2.3) and (2.4); (b) for fixed  $p$  and  $\mu$ , as a bounded from above function,  $E(y, p, \mu)$  has global maxima each of which is also its local maximum.

To get (b) we observe that (2.17) implies  $\lim_{|y| \rightarrow +\infty} E(y, p, \mu) = -\infty$ ; hence, each point  $\bar{y}$  of global maximum belongs to a certain interval  $(\bar{y} - \varepsilon, \bar{y} + \varepsilon)$ , where it is also a maximum point. Since  $E$  is everywhere differentiable in  $y$ , then  $\bar{y}$  is the point of global maximum only if it solves the following equation

$$E_1(y, p, \mu) := \frac{\partial}{\partial y} E(y, p, \mu) = 0. \quad (2.18)$$

By (2.14) and (2.15) this equation can be rewritten in the form

$$-\frac{y}{p} + \frac{K_1(y, p, \mu)}{K(y, p, \mu)} = 0, \quad (2.19)$$

$$K_1(y, p, \mu) := \sum_{n=1}^{\infty} \frac{nv^n}{n!} \exp\left((y + \mu)n - \frac{ap}{2}n^2\right).$$

*Remark 2.2.* As we will see from the proof of Theorem 2.9 below, the equation in (2.19) has at least one solution for all  $p > 0$  and  $\mu \in \mathbb{R}$ . Since both  $K_1$  and  $K$  take only strictly positive values, these solutions are also strictly positive.

**Definition 2.3.** We say that  $(p, \mu)$  belongs to a single-phase domain if  $E(y, p, \mu)$  has a unique global maximum  $\bar{y} \in \mathbb{R}$  such that

$$E_2(\bar{y}, p, \mu) := \frac{\partial^2}{\partial y^2} E(y, p, \mu)|_{y=\bar{y}} < 0. \quad (2.20)$$

Note that  $\bar{y}$  can be a point of maximum if  $E_1(\bar{y}, p, \mu) = E_2(\bar{y}, p, \mu) = 0$ . That is, not every point of global maximum corresponds to a point in a single-phase domain.

The condition in (2.19) determines the unique probability measure  $Q_{p,\mu}$  on  $\mathbb{N}_0$  such that

$$Q_{p,\mu}(n) = \frac{1}{K(\bar{y}, p, \mu)n!} v^n \exp\left((\bar{y} + \mu)n - \frac{ap}{2}n^2\right), \quad n \in \mathbb{N}_0, \quad (2.21)$$

which yields the probability law of the occupation number of a single cell. Then the unique thermodynamic phase of the model corresponding to  $(p, \mu) \in \mathcal{R}$  is the product

$$\mathbf{Q}_{p,\mu} = \bigotimes_{\ell=1}^{\infty} Q_{p,\mu}^{(\ell)} \quad (2.22)$$

of the copies of the measure defined in (2.21). It is a probability measure on the space of all vectors  $\mathbf{n} = (n_\ell)_{\ell=1}^{\infty}$ , in which  $n_\ell \in \mathbb{N}_0$  is the occupation number of  $\ell$ -th cell.

The role of the condition in (2.20) is to yield the possibility to apply Laplace's method for asymptotic calculating the integral in (2.13). By direct calculations it follows that

$$\begin{aligned} E_2(y, p, \mu) &= -\frac{1}{p} + \frac{1}{2[K(y, p, \mu)]^2} \\ &\times \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} (n_1 - n_2)^2 \exp\left((y + \mu)(n_1 + n_2) - \frac{ap}{2}(n_1^2 + n_2^2)\right). \end{aligned} \quad (2.23)$$

In dealing with the equation in (2.19) we will fix  $p > 0$  and consider  $E_1$  as a function of  $y \in \mathbb{R}$  and  $\mu \in \mathbb{R}$ . Then, for a given  $\mu$ , we solve (2.17) to find  $\bar{y}$  and then check whether it is the unique point of global maximum and (2.20) is satisfied, i.e., whether  $(p, \mu)$  belongs to a single-phase domain. Then we slightly vary  $\mu$  and repeat the same. This will yield a function  $\mathbb{R} \ni \mu \mapsto \bar{y}(\mu)$ , dependent on the choice of  $p$ . In doing so, we will use the analytic implicit function theorem based on the fact that, for each fixed  $p > 0$ , the function  $\mathbb{R}^2 \ni (y, \mu) \mapsto E_1(y, p, \mu)$  can be analytically continued to a some complex neighborhood of  $\mathbb{R}^2$ , see (2.15) and (2.19). For reader's convenience, we present this theorem here in the form adapted from [11, Section 7.6, page 34]. For some  $p_0 > 0$ , let

$\mathcal{B} \subset \mathbb{C}^2$  be a connected open set containing  $\mathbb{R}^2$  such that the function  $(y, \mu) \mapsto E_1(y, p_0, \mu)$  is analytic in  $\mathcal{B}$ .

**Proposition 2.4** (Implicit function theorem). *Let  $p_0$  and  $(y_0, \mu_0)$  be such that  $E_1(y_0, p_0, \mu_0) = 0$  and  $E_2(y_0, p_0, \mu_0) \neq 0$ . Let also  $\mathcal{B} \subset \mathbb{C}^2$  be as just described. Then there exist open sets  $\mathcal{D}_1 \subset \mathbb{C}$  and  $\mathcal{D}_2 \subset \mathbb{C}$  such that  $y_0 \in \mathcal{D}_1$ ,  $\mu_0 \in \mathcal{D}_2$ ,  $\mathcal{D}_1 \times \mathcal{D}_2 \subset \mathcal{B}$ , and an analytic function  $\bar{y} : \mathcal{D}_2 \rightarrow \mathcal{D}_1$ , for which the following holds*

$$\{(y, \mu) \in \mathcal{D}_1 \times \mathcal{D}_2 : E_1(y, p_0, \mu) = 0\} = \{(\bar{y}(\mu), \mu) : \mu \in \mathcal{D}_2\}.$$

The derivative of  $\bar{y}$  in  $\mathcal{D}_2$  is

$$\frac{d\bar{y}(\mu)}{d\mu} = -\frac{1}{E_2(\bar{y}(\mu), p_0, \mu)} \left[ \frac{\partial}{\partial \mu} E_1(y, p_0, \mu) \right]_{y=\bar{y}(\mu)}. \quad (2.24)$$

*Remark 2.5.* In the sequel, we will use also the version of the implicit function theorem in which we do not employ the analytic continuation of  $E_1$  to complex values of  $p$ . Let the conditions of Proposition 2.4 regarding  $E_1$  and  $E_2$  be satisfied. Then there exist open sets  $\mathcal{D}_i \subset \mathbb{R}$ ,  $i = 1, 2, 3$ , and a continuous function  $\bar{y} : \mathcal{D}_3 \times \mathcal{D}_2 \rightarrow \mathcal{D}_1$  such that  $p_0 \in \mathcal{D}_3$ ,  $\mu_0 \in \mathcal{D}_2$  and the following holds

$$\{(y, p, \mu) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 : E_1(y, p, \mu) = 0\} = \{(\bar{y}(p, \mu), p, \mu) : p \in \mathcal{D}_3, \mu \in \mathcal{D}_2\}.$$

The partial derivative of  $\bar{y}(p, \mu)$  over  $\mu \in \mathcal{D}_2$  is given by the right-hand side of (2.24). In the sequel, by writing  $\bar{y}(\mu)$  we assume both the function as Proposition 2.4, defined for a fixed  $p$  known from the context, and that as in Remark 2.5 with the fixed value of  $p$ .

For a fixed  $p_0 > 0$ , assume that  $(p_0, \mu_0)$  belongs to a single-phase domain. By Proposition 2.4 there exists  $\varepsilon > 0$  such that the function  $(\mu_0 - \varepsilon, \mu_0 + \varepsilon) \ni \mu \mapsto \bar{y}(\mu)$  can be defined by the equation  $E_1(y, p_0, \mu) = 0$ . Its continuation from the mentioned interval is related to the fulfilment of the condition  $E_2(\bar{y}(\mu), p_0, \mu) < 0$ , cf. (2.20), which may not be the case. At the same time, by (2.23) we have that

$$\frac{\partial}{\partial \mu} E_1(y, p, \mu) = E_2(y, p, \mu) + \frac{1}{p} > 0,$$

holding for all  $y \in \mathbb{R}$ ,  $p > 0$  and  $\mu \in \mathbb{R}$ . In view of this and (2.24), it might be more convenient to use the inverse function  $y \mapsto \bar{\mu}(y)$  since the  $\mu$ -derivative of  $E_1$  is always nonzero. Its properties are described by the following statement obtained from the analytic implicit function theorem mentioned above. Recall that only positive  $y$  solve the equation in (2.19).

**Proposition 2.6.** *Given  $p_0$ , let  $\mathcal{B}$  be as in Proposition 2.4. Then there exists open connected subsets  $\mathcal{D}_i \subset \mathbb{C}$ ,  $i = 1, 2$ , and an analytic function  $\mathcal{D}_1 \ni y \mapsto \bar{\mu}(y) \in \mathcal{D}_2$  such that  $\mathcal{D}_1$  contains  $\mathbb{R}_+$ ,  $\mathcal{D}_1 \times \mathcal{D}_2 \subset \mathcal{B}$ , and the following holds*

$$\{(y, \mu) \in \mathcal{D}_1 \times \mathcal{D}_2 : E_1(y, p_0, \mu) = 0\} = \{(y, \bar{\mu}(y)) : y \in \mathcal{D}_1\}.$$

The derivative of  $\bar{\mu}$  in  $\mathcal{D}_1$  is

$$\frac{d\bar{\mu}(y)}{dy} = -\frac{E_2(y, p_0, \bar{\mu}(y))}{E_2(y, p_0, \bar{\mu}(y)) + \frac{1}{p}}.$$

**Proposition 2.7.** *Each single-phase domain,  $\mathcal{R}$ , has the following properties: (a) it is an open subset of  $\mathbb{R}_+ \times \mathbb{R}$ ; (b) for each  $(p_0, \mu_0) \in \mathcal{R}$ , the function  $\mathcal{I}_{p_0} := \{\mu \in \mathbb{R} : (p_0, \mu) \in \mathcal{R}\} \ni \mu \mapsto \bar{y}(\mu)$  as in Proposition 2.4 is continuously differentiable on  $\mathcal{I}_{p_0}$ . Moreover,*

$$\frac{d\bar{y}(\mu)}{d\mu} > 0, \quad \text{for all } \mu \in \mathcal{I}_{p_0}. \quad (2.25)$$

*Proof.* For a single-phase domain,  $\mathcal{R}$ , take  $(p_0, \mu_0) \in \mathcal{R}$ . By Remark 2.5 the function  $(p, \mu) \mapsto \bar{y}(p, \mu)$ , defined by the equation  $E_1(y, p, \mu) = 0$  is continuous in some open subset of  $\mathcal{R}$  containing  $(p_0, \mu_0)$ . By the continuity of  $E_2(\bar{y}(p, \mu), p, \mu)$  and the fact that  $E_2(\bar{y}(p_0, \mu_0), p_0, \mu_0) < 0$  (since  $(p_0, \mu_0) \in \mathcal{R}$ ) we get that  $E_2(\bar{y}(p, \mu), p, \mu) < 0$  for  $(p, \mu)$  in some open neighborhood of  $(p_0, \mu_0)$ . Hence,  $\mathcal{R}$  contains  $(p_0, \mu_0)$  with some neighborhood and thus is open. The continuous differentiability of  $\bar{y}$  and the sign rule in (2.25) follow by Proposition 2.4 and (2.24), respectively.  $\square$

By (2.21) and (2.19) we get the  $Q_{p,\mu}$ -mean value  $\bar{n} = \bar{n}(p, \mu)$  of the occupation number of a given cell in the form

$$\bar{n}(p, \mu) = \sum_{n=0}^{\infty} n Q_{p,\mu}(n) = \frac{K_1(\bar{y}(p, \mu), p, \mu)}{K(\bar{y}(p, \mu), p, \mu)} = \frac{\bar{y}(p, \mu)}{p}. \quad (2.26)$$

Note that, up to the factor  $v^{-1}$ ,  $\bar{n}(p, \mu)$  is the particle density in phase  $\mathbf{Q}_{p,\mu}$ . For a fixed  $p$ , by Proposition 2.7  $\bar{n}(p, \cdot)$  is an increasing function on  $\mathcal{I}_p$ , which thus can be inverted to give  $\bar{\mu}(p, \bar{n})$ . By Laplace's method we then get the following corollary of Proposition 2.7.

**Proposition 2.8.** *Let  $\mathcal{R}$  be a single-phase domain. Then, for each  $(p, \mu) \in \mathcal{R}$ , the limiting pressure  $P(p, \mu) = \lim_{N \rightarrow +\infty} P_N(p, \mu)$ , see (2.16), exists and is continuously differentiable on  $\mathcal{R}$ . Moreover, it is given by the following formula*

$$P(p, \mu) = v^{-1} E(\bar{y}(p, \mu), p, \mu). \quad (2.27)$$

Let  $\mathcal{N}_p$  be the image of  $\mathcal{I}_p$  under the map  $\mu \mapsto \bar{n}(p, \mu)$ . Then the inverse map  $\bar{n} \mapsto \bar{\mu}(p, \bar{n})$  is continuously differential and increasing on  $\mathcal{N}_p$ . By means of this map, for a fixed  $p$ , the pressure given in (2.27) can be written as a function of  $\bar{n}$

$$P = \bar{P}(\bar{n}) = v^{-1} E(p\bar{n}, p, \bar{\mu}(p, \bar{n})), \quad \bar{n} \in \mathcal{N}_p, \quad (2.28)$$

which is the equation of state.

**2.4. The phase transition.** Recall that the notion of the single-phase domain was introduced in Definition 2.3, and each such a domain is an open subset of the open right half-plane  $\{(p, \mu) : p > 0, \mu \in \mathbb{R}\}$ , see Proposition 2.7. With this regard we have the following possibilities: (i) the whole half-plane  $\{(p, \mu) : p > 0, \mu \in \mathbb{R}\}$  is such a domain; (ii) there exist more than one single-phase domains. In case (i), for all  $(p, \mu)$  there exists one phase (2.22). In the context of this work, a phase transition is understood as the possibility to have different phases at the same value of the pair  $(p, \mu)$ . If this is the case,  $(p, \mu)$  is called a *phase coexistence point*. Clearly, such a point should belong to the common topological boundary of at least two distinct single-phase domains. That is, to prove the existence of phase transitions we have to show that possibility (ii) takes place and that these single-phase domains have common boundary. We do this in Theorems 2.9 and 2.10 below.

Let  $\mathcal{R}$  be a single-phase domain. Take  $(p_0, \mu_0) \in \mathcal{R}$  and consider the line  $l_{p_0} = \{(p_0, \mu) : \mu \in \mathbb{R}\}$ . If the whole line lies in  $\mathcal{R}$ , by Proposition 2.7  $\bar{y}(\mu)$  is a continuously differentiable and increasing function of  $\mu \in \mathbb{R}$ . In our first theorem, we prove that this is the case for small enough  $p_0$ .

**Theorem 2.9.** *There exists  $p_0 > 0$  such that the set  $\mathcal{R}(p_0) := \{(p, \mu) : p \in (0, p_0]\}$  is a single-phase domain.*

*Proof.* In view of Remark 2.1, we have to show that, for fixed  $p \leq p_0$  and all  $\mu \in \mathbb{R}$ ,  $E(y, p, \mu)$  has exactly one local maximum such that (2.20) holds. For  $x \in \mathbb{R}$ , we set, cf.



(2.15),

$$\phi(x, p) = \ln \sum_{n=0}^{\infty} \frac{v^n}{n!} \exp \left( xn - \frac{ap}{2} n^2 \right), \quad (2.29)$$

$$\phi_k(x, p) = \frac{\partial^k}{\partial x^k} \phi(x, p), \quad k = 1, 2.$$

Similarly as in (2.17) we then get

$$\phi(x, p) \leq v + \frac{x^2}{2ap}. \quad (2.30)$$

Note also that, for the functions defined in (2.29), we have

$$\lim_{p \rightarrow 0} \phi(x, p) = \lim_{p \rightarrow 0} \phi_1(x, p) = \lim_{p \rightarrow 0} \phi_2(x, p) = ve^x, \quad x \in \mathbb{R}. \quad (2.31)$$

By means of (2.29) we rewrite the equation in (2.19) in the following form

$$\begin{cases} x = y + \mu, \\ \mu = x - p\phi_1(x, p). \end{cases} \quad (2.32)$$

Our aim is to show that there exists  $p_0 > 0$  such that, for each  $p \in (0, p_0]$ , the following holds: (a) the second line in (2.32) defines an increasing unbounded function  $\tilde{\mu}(x)$ ,  $x \in \mathbb{R}$ ; (b)  $p\phi_2(x, p) \leq 1 - \delta$  for some  $\delta \in (0, 1)$  and all  $x \in \mathbb{R}$ . Indeed, the function mentioned in (a) can be inverted to give an unbounded increasing function  $\tilde{x}(\mu)$ ,  $\mu \in \mathbb{R}$ , such that the solution of (2.19) is  $\bar{y}(\mu) = \tilde{x}(\mu) - \mu$ . Then by (2.29) and (2.23) we get

$$E_2(\bar{y}(\mu), p, \mu) = -\frac{1}{p} + \phi_2(\tilde{x}(\mu), p) < 0,$$

where the latter inequality follows by (b). Thus, to prove both (a) and (b) it is enough to show that there exists positive  $p_0$  such that

$$p\phi_2(x, p) \leq \frac{1}{a}, \quad \text{for all } x \in \mathbb{R} \text{ and } p \in (0, p_0]. \quad (2.33)$$

By (2.29) we have

$$\phi_2(x, p) = \frac{1}{2\Phi(x, p)} \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} (n_1 - n_2)^2 \quad (2.34)$$

$$\times \exp \left( x(n_1 + n_2) - \frac{ap}{2} (n_1^2 + n_2^2) \right),$$

$$\Phi(x, p) = \left( \sum_{n=0}^{\infty} \frac{v^n}{n!} \exp \left( xn - \frac{ap}{2} n^2 \right) \right)^2 \quad (2.35)$$

$$= \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} \exp \left( x(n_1 + n_2) - \frac{ap}{2} (n_1^2 + n_2^2) \right).$$

Since  $\Phi(x, p) \geq 1$ , we get from the latter

$$\begin{aligned}
\phi_2(x, p) &\leq \frac{1}{2} \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} (n_1 - n_2)^2 \exp \left( x(n_1 + n_2) - \frac{ap}{2}(n_1^2 + n_2^2) \right) \\
&\leq \frac{1}{2} \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} (n_1 - n_2)^2 \exp \left( x(n_1 + n_2) \right) \\
&= \left( \sum_{n=1}^{\infty} \frac{n^2}{n!} (ve^x)^n \right) \cdot \left( \sum_{n=0}^{\infty} \frac{1}{n!} (ve^x)^n \right) - \left( \sum_{n=1}^{\infty} \frac{n}{n!} (ve^x)^n \right)^2 \\
&= v \exp \left( x + 2ve^x \right).
\end{aligned} \tag{2.36}$$

Fix some  $x_0 > 0$  and then set

$$p_{01} = \frac{1}{av} \exp \left( -x_0 - 2ve^{x_0} \right). \tag{2.37}$$

Then by (2.36) we get

$$p\phi_2(x, p) \leq \frac{1}{a}, \quad \text{for all } x \leq x_0 \text{ and } p \leq p_{01}. \tag{2.38}$$

By (2.29) and (2.31) we see that the function

$$\psi(p) = \frac{v - \phi(0, p)}{\phi_1(0, p)} \tag{2.39}$$

continuously depends on  $p > 0$  and  $\psi(p) \rightarrow 0$  as  $p \rightarrow 0$ . For each  $x > 0$ , one finds  $\xi \in (0, 1]$ , dependent on  $x$  and  $p$ , such that

$$\phi(x, p) = \phi(0, p) + \phi_1(0, p)x + \frac{1}{2}\phi_2(\xi x, p)x^2.$$

From this and (2.30) we get

$$\phi(\xi^{-1}x, p) = \phi(0, p) + \phi_1(0, p)\xi^{-1}x + \frac{1}{2}\phi_2(x, p)\xi^{-2}x^2 \leq v + \frac{\xi^{-2}x^2}{2ap}. \tag{2.40}$$

For the function defined in (2.39) and  $x_0$  as in (2.37), we pick  $p_{02}$  such that  $\psi(p) \leq x_0$  for all  $p \leq p_{02}$ . For such values of  $p$ , this yields

$$\phi_1(0, p)\xi^{-1}x \geq \phi_1(0, p)x_0 \geq v - \phi(0, p), \quad \text{for all } x \geq x_0.$$

We apply this in (2.40) and get

$$p\phi_2(x, p) \leq \frac{1}{a}, \quad \text{for all } x \geq x_0 \text{ and } p \leq p_{02}. \tag{2.41}$$

This and (2.38) then yields (2.33) with  $p_0 = \min\{p_{01}; p_{02}\}$ , which completes the proof.  $\square$

**Theorem 2.10.** *For each  $a > 1$ , there exists  $p_1 = p_1(a) > 0$  such that, for each  $p \geq p_1$ , the line  $l_p = \{(p, \mu) : \mu \in \mathbb{R}\}$  contains at least one phase-coexistence point.*

*Proof.* Let  $\bar{\mu}(y)$  be the function as in Proposition 2.6. By (2.29) and (2.32) we have that

$$\bar{\mu}(y) = \left[ x - p\phi_1(x, p) \right]_{x=\bar{x}(y)}, \tag{2.42}$$

and

$$\lim_{y \rightarrow 0} \bar{\mu}(y) = -\infty, \quad \lim_{y \rightarrow +\infty} \bar{\mu}(y) = +\infty.$$

In (2.42),  $\bar{x}(y)$  is the inverse of the function  $\mathbb{R} \ni x \mapsto y = p\phi_1(x, p)$ . Note that

$$\frac{d\bar{x}(y)}{dy} = \left[ \frac{1}{p\phi_2(x, p)} \right]_{x=\bar{x}(y)}, \quad (2.43)$$

hence  $(0, +\infty) \ni y \mapsto \bar{x}(y) \in \mathbb{R}$  is increasing. By (2.42) and (2.43) it follows that

$$\frac{d\bar{\mu}(y)}{dy} = \frac{d\bar{x}(y)}{dy} \left[ 1 - p\phi_2(x, p) \right]_{x=\bar{x}(y)} = \left[ \frac{1}{p\phi_2(x, p)} \right]_{x=\bar{x}(y)} - 1. \quad (2.44)$$

For a given  $p > 0$ , pick  $x^p > 0$  such that  $\psi(p)$  defined in (2.39) satisfies  $\psi(p) \leq x^p$ . Then as in (2.41) we obtain  $p\phi_2(x, p) \leq 1/a$  for all  $x \geq x^p$ . By (2.44) this yields that

$$\frac{d\bar{\mu}(y)}{dy} \geq a - 1, \quad \text{for } y \in [y^p, +\infty), \quad (2.45)$$

where  $y^p = p\phi_1(x^p, p)$ . For the same  $p$ , let  $x_p$  be such that  $avp = \exp(-x_p - 2ve^{x_p})$ , cf. (2.37). Then by (2.36) and (2.44) we conclude that the inequality in (2.45) holds also for  $y \in (0, y_p]$ ,  $y_p := p\phi_1(x_p, p)$ . As we have seen in the proof of Theorem 2.9, the mentioned two intervals may overlap, i.e., it may be that  $y_p > y^p$ , if  $p$  is small enough. Let us prove that this is not the case for big  $p$ . That is, let us show that there exists  $p_1 > 0$  such that, cf. (2.44) and (2.41), for all  $p \geq p_1$ , there exists  $x \in \mathbb{R}$  such that following holds

$$p_1\phi_2(x, p_1) \geq 1, \quad \text{and } p\phi_2(x, p) > 1, \quad \text{for } p > p_1. \quad (2.46)$$

To this end we estimate the denominator of (2.34) from above and the numerator from below. For  $x = ap/2$ , by (2.35) it follows that

$$\Phi(x, p) = \left( \sum_{n=0}^{\infty} \frac{v^n}{n!} \exp \left[ -\frac{ap}{2} n(n-1) \right] \right)^2 \leq \left( \sum_{n=0}^{\infty} \frac{v^n}{n!} \right)^2 = e^{2v}, \quad (2.47)$$

where we used the fact that  $p > 0$ . In the sum in the numerator of (2.34), we take just two summands corresponding to  $n_1 = 1, n_2 = 0$  and  $n_1 = 0, n_2 = 1$ , and obtain by (2.47) the following estimate

$$p\phi_2(x, p) \geq pve^{-2v}, \quad (2.48)$$

holding for  $x = ap/2$ . Then we set  $p_1 = v^{-1}e^{2v}$ . For this  $p_1$  and  $x = ap_1/2$ , by (2.48) we obtain (2.46). Clearly, for  $p > p_1$ ,  $x_p$  and  $x^p$  introduced above satisfy

$$x_p < ap/2 < x^p.$$

For  $p > p_1$ , let  $(x_p, x^p)$  be the biggest interval which contains  $x = ap/2$  and is such that  $p\phi_2(x, p) > 1$  for each  $x \in (x_p, x^p)$ . Then  $p\phi_2(x, p) = 1$  for  $x = x_p$  and  $x = x^p$ . Set

$$y_p = p\phi_1(x_p, p), \quad y^p = p\phi_1(x^p, p).$$

Then by (2.44) and (2.45) it follows that

$$\frac{d\bar{\mu}(y)}{dy} = 0, \quad \text{for } y = y_p, y^p, \quad (2.49)$$

$$\frac{d\bar{\mu}(y)}{dy} < 0, \quad \text{for } y \in (y_p, y^p),$$

$$\frac{d\bar{\mu}(y)}{dy} \geq a - 1 > 0, \quad \text{for } y < y_p \text{ and } y > y^p.$$

From this we see that  $y_p$  (resp.  $y^p$ ) is the first maximum (resp. the last minimum) of  $\bar{\mu}(y)$ . Let  $\hat{y}^p$  be the first minimum of  $\bar{\mu}(y)$ . Set  $\hat{\mu}^p = \bar{\mu}(\hat{y}^p)$ . Now we pick  $y_2 > \hat{y}^p$  such that: (a) either  $\bar{\mu}(y_2) = \bar{\mu}(y_p)$ ; (b) or  $y_2$  is the second maximum of  $\bar{\mu}(y)$  if  $\bar{\mu}(y_2) \leq \bar{\mu}(y_p)$ . Then set  $\hat{\mu}_p = \bar{\mu}(y_2)$ . Clearly,  $\hat{\mu}_p > \hat{\mu}^p$ . In case (a), we have  $\hat{\mu}_p = \bar{\mu}(y_p)$ ; and  $\hat{\mu}_p \leq \bar{\mu}(y_p)$  in case (b). Now we pick  $y_1 \in (0, y_p)$  such that  $\bar{\mu}(y_1) = \bar{\mu}(\hat{y}^p)$ . By (2.49) and the above

construction the function  $\bar{\mu}(y)$  is increasing on  $[y_1, y_p)$  and  $(\hat{y}_p, y_2)$ , and decreasing on  $(y_p, \hat{y}^p)$ , see Figure 2. Let  $\bar{y}_1$  and  $\bar{y}_2$  be the inverse functions to the restrictions of  $\bar{\mu}(y)$  to  $[y_1, y_p)$  and  $(\hat{y}_p, y_2)$ , respectively. Let also  $\bar{y}_3$  be the inverse function to the restrictions of  $\bar{\mu}(y)$  to the interval  $(y_p, \hat{y}^p)$ . All the three functions are defined on  $M_p := (\hat{\mu}^p, \hat{\mu}_p)$  and are continuously differentiable thereon. Note that  $\bar{y}_2(\mu) > \bar{y}_3(\mu) > \bar{y}_1(\mu)$  for all  $\mu \in (\hat{\mu}^p, \hat{\mu}_p)$  and

$$\bar{y}_1(\hat{\mu}_p) = \bar{y}_3(\hat{\mu}_p), \quad \bar{y}_2(\hat{\mu}^p) = \bar{y}_3(\hat{\mu}^p). \quad (2.50)$$

Moreover, all the three  $\bar{y}_i(\mu)$ ,  $i = 1, 2, 3$ , satisfy (2.19) and, for  $\mu \in (\hat{\mu}^p, \hat{\mu}_p)$ ,  $E(y, p, \mu)$  has local maxima at  $y = \bar{y}_i(\mu)$ ,  $i = 1, 2$  and a local minimum at  $y = \bar{y}_3(\mu)$ . This follows from the fact that  $E_2(\bar{y}_i(\mu), p, \mu) < 0$  for  $\mu \in M_p$  and  $i = 1, 2$ , and from  $E_2(\bar{y}_3(\mu), p, \mu) > 0$  for  $\mu \in M_p$ , see (2.49).

Set

$$D(\mu) = E(\bar{y}_2(\mu), p, \mu) - E(\bar{y}_1(\mu), p, \mu), \quad \mu \in M_p. \quad (2.51)$$

If  $D(\mu) < 0$ , then  $\bar{y}_1(\mu)$  is the point of global maximum of  $E(y, p, \mu)$  and hence  $(p, \mu)$  lies in a single-phase domain, say  $\mathcal{R}_1$ . If  $D(\mu) > 0$ , then the same holds for  $\bar{y}_2(\mu)$  and  $\mathcal{R}_2$ . If

$$D(\mu_1) < 0, \quad \text{and} \quad D(\mu_2) > 0, \quad (2.52)$$

for some  $\mu_1, \mu_2 \in M_p$ , then there should exist  $\mu_c$  in between where  $D$  vanishes. Thus,  $(p, \mu_c)$  belongs to the boundaries of both  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , and hence is a phase coexistence point, if  $\mu_c$  is an isolated zero of (2.51). The phases are then given in (2.21) and (2.22) with  $\bar{y}_1(\mu_c)$  and  $\bar{y}_2(\mu_c)$ , respectively. Note that the vanishing of  $D$  at  $\mu_c$  corresponds to the Maxwell rule, cf. [1], and to the existence of two global maxima of  $E(y, p, \mu)$ . Since both  $\bar{y}_i(\mu)$  are differentiable, by (2.14), (2.15), (2.18), and (2.23) we have

$$\begin{aligned} \frac{dD(\mu)}{d\mu} &= E_1(\bar{y}_2(\mu), p, \mu) \frac{d\bar{y}_2(\mu)}{d\mu} + \frac{K_1(\bar{y}_2(\mu), p, \mu)}{K(\bar{y}_2(\mu), p, \mu)} \\ &\quad - E_1(\bar{y}_1(\mu), p, \mu) \frac{d\bar{y}_1(\mu)}{d\mu} - \frac{K_1(\bar{y}_1(\mu), p, \mu)}{K(\bar{y}_2(\mu), p, \mu)} \\ &= \frac{K_1(\bar{y}_2(\mu), p, \mu)}{K(\bar{y}_2(\mu), p, \mu)} - \frac{K_1(\bar{y}_1(\mu), p, \mu)}{K(\bar{y}_2(\mu), p, \mu)} \\ &= E_2(\bar{y}_*(\mu), p, \mu) + \frac{1}{p} > 0 \end{aligned}$$

for some  $\bar{y}_*(\mu) \in [\bar{y}_1(\mu), \bar{y}_2(\mu)]$ . Note that  $E_1(\bar{y}_i(\mu), p, \mu) = 0$ ,  $i = 1, 2$ , cf. (2.18). Therefore,  $D$  can hit the zero level at most once. Let us show that (2.52) does hold. If  $\mu_1 \in M_p$  is close enough to  $\hat{\mu}^p$ , then by (2.50) and the mentioned continuity we have that  $E(\bar{y}_2(\mu_1), p, \mu_1)$  is close to  $E(\bar{y}_3(\mu_1), p, \mu_1)$ , and hence  $\bar{y}_2$  cannot be the global maximum of  $E(y, p, \mu_1)$ . Therefore,  $D(\mu_1) < 0$  for such  $\mu$ . Likewise we establish the existence of  $\mu_2$  such that  $D(\mu_2) > 0$ . Now the existence of  $\mu_c \in (\mu_1, \mu_2)$  follows by the continuity and (2.52).  $\square$

### 3. NUMERICAL RESULTS

Here we present the results of numerical calculations of the functions which appear in the preceding part of the paper.

We begin by considering the extremum points of the functions which appear in Section 2.4. According to Definition 2.3, the line  $l_p = \{(p, \mu) : \mu \in \mathbb{R}\}$  lies in a single-phase domain, if the function  $\mathbb{R}_+ \ni y \mapsto E(y, p, \mu)$ , see (2.14), has a unique non-degenerate global maximum for all  $\mu \in \mathbb{R}$ . The corresponding condition in (2.19) determines an increasing function  $\bar{y}(\mu)$ , see Proposition 2.7, which can be inverted to give  $\bar{\mu}(y)$ , see (2.42). In Theorem 2.9, we show that this holds for sufficiently small  $p$ . Figure 1 presents the results of the calculation of  $\bar{\mu}(y)$  for

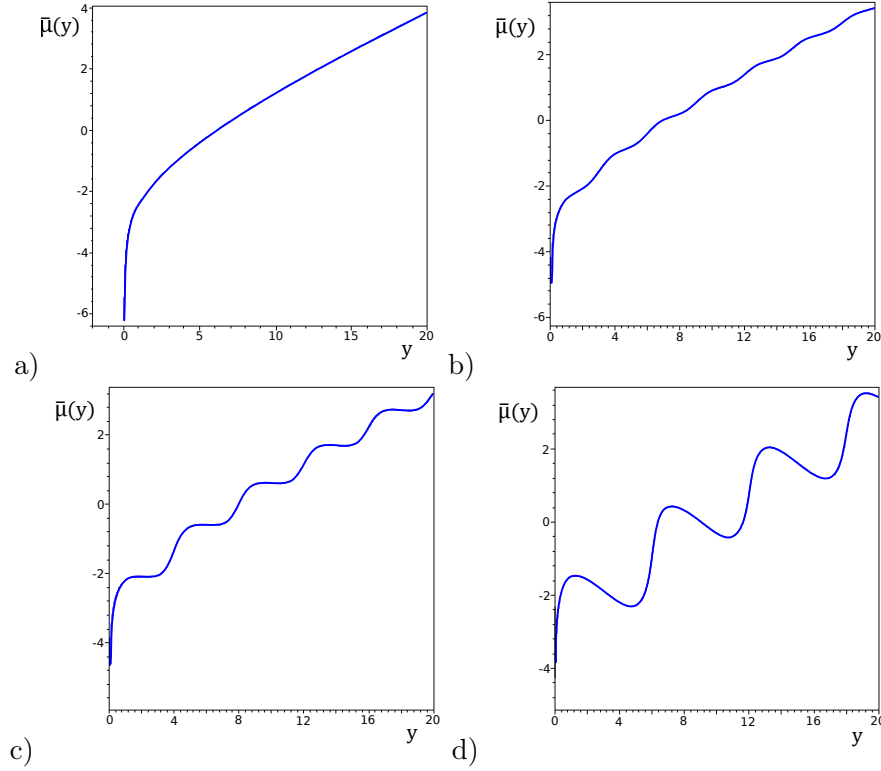


FIGURE 1. Plot of  $\bar{\mu}(y)$  defined in (2.42) for  $a = 1.2$ ,  $v = 12$  and various values of the attraction parameter:  $p = 2$  (curve a),  $p = 3$  (curve b),  $p = 4$  (curve c),  $p = 6$  (curve d).

$$a = 1.2, \quad v = 12, \quad (3.1)$$

and  $p = 2, 3, 4, 6$  — curves (a), (b), (c) and (d), respectively. In the first two cases  $\bar{\mu}$  is an increasing function, which corresponds to the situation described in Theorem 2.9. That is, the values of  $p = 2$  and  $3$  are below the critical value  $p_c = p_c(a)$ . For  $a$  and  $v$  as in (3.1), our calculations yield

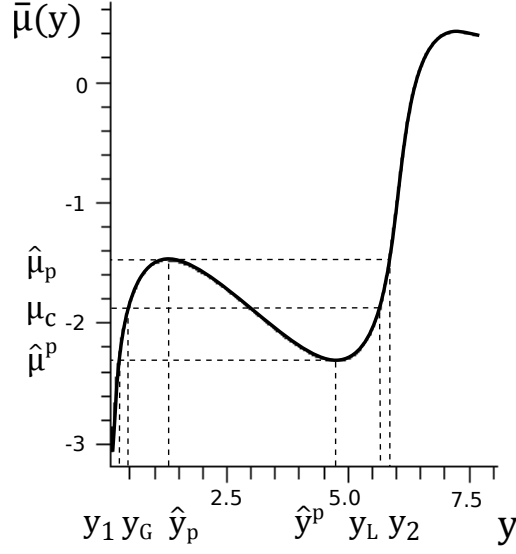
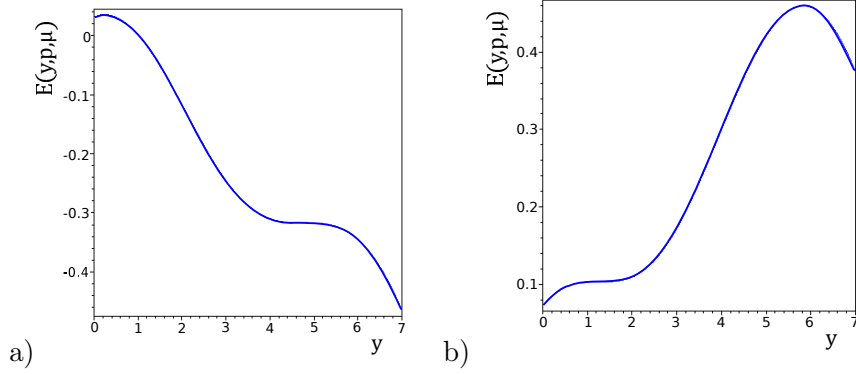
$$p_c = 3.928235(8).$$

For  $p = p_c$ , the function  $y \mapsto E(y, p_c, \mu)$  still has a unique global maximum, which gets degenerate, i.e.,  $E_2(\bar{y}, p_c, \mu) = 0$ , cf. (2.20). The value of  $\bar{y} = \bar{y}_c$  at which this occurs gives the value of the critical density  $\bar{n}_c = \bar{y}_c/p_c$ , see (2.28). For various values of the parameter  $a$ , see (2.4), the values of  $p_c(a)$ ,  $\bar{y}_c(a)$  and  $\bar{n}_c(a)$  are given in following table.

TABLE 1. Values of  $p(a)$ ,  $\bar{y}(a)$ ,  $\bar{n}(a)$  in the critical point for  $v = 12$

$a$	1.0001	1.2	1.5	2	10
$p_c(a)$	3.8255	3.9282	3.9796	3.9973	4.0000
$\bar{y}_c(a)$	2.0485	2.0187	2.0052	2.0007	2.0000
$\bar{n}_c(a)$	0.5355	0.5139	0.5038	0.5005	0.5000

The values of  $p = 4$  and  $6$  are above the critical point  $p_c$ , which can clearly be seen from the curves (c) and (d) of Figure 1. In this case, one deals with the situation described by Theorem 2.10. Figure 2 presents in more detail the curve plotted in Figure 1 (d), i.e., corresponding to  $p = 6$  and  $a = 1.2$ . It provides a good illustration to the proof of Theorem 2.10. Here we have  $\bar{\mu}(y_2) = \bar{\mu}(y_p)$ .

FIGURE 2. Plot of  $\bar{\mu}(y)$  for  $a = 1.2$  and  $p = 6$  in more detail.FIGURE 3. Plot of the function  $E(y, p, \mu)$  for  $p = 6$ ,  $a = 1.2$  and  $\mu_1 = -2.3080$  (curve a),  $\mu_2 = -1.4700$  (curve b).

Let us now turn to the maximum points of  $E(y, p, \mu)$ . For  $a$  as in (3.1) and  $p = 6$ , Figure 3 presents the dependence of  $E$  on  $y$  for  $\mu_1 = -2.3080$  (curve a), and  $\mu_2 = -1.4700$  (curve b). This provides an illustration to (2.52). The curve plotted in Figure 4 corresponds to the critical value of  $\mu$  defined by the condition  $D(\mu) = 0$ . That is,  $(p, \mu_c)$  with  $p = 6$  and  $\mu_c = -1.890291$  is the phase coexistence point the existence of which was proved in Theorem 2.10. Figure 5 presents the dependence of  $E(\bar{y}_1(\mu), p, \mu)$  (line 1, red) and  $E(\bar{y}_2(\mu), p, \mu)$  (line 2, blue) on  $\mu \in M_p$ ,  $p = 6$ , cf. (2.51). Their intersection occurs at  $\mu = \mu_c = -1.890291$ .

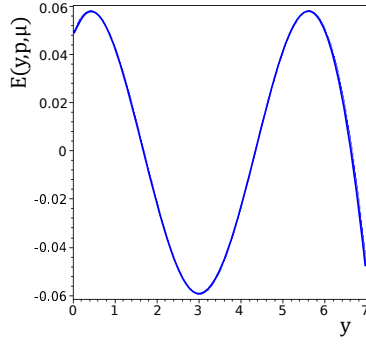


FIGURE 4. The same as in Figure 3 for  $\mu = \mu_c = -1.890291$ .

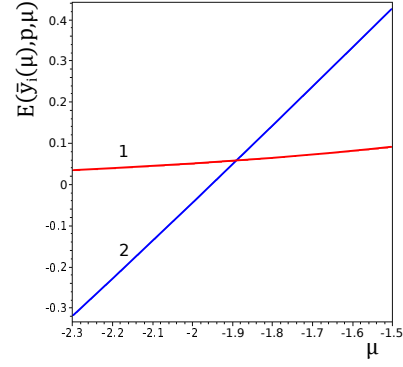


FIGURE 5. Plot of the functions  $M_p \ni \mu \mapsto E(y_i(\mu), p, \mu)$ ,  $i = 1, 2$ , see (2.51), and  $p = 6$ ,  $a = 1.2$ .

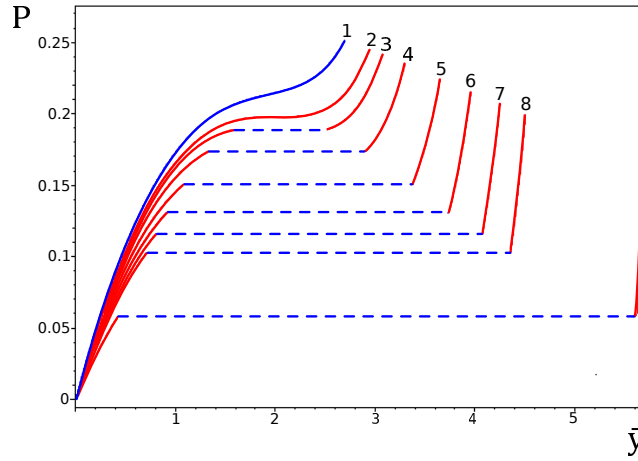


FIGURE 6. Plot of the dependence of the pressure on  $\bar{y} = p\bar{n}$  (isotherms) see (2.27) and (2.28). Curve 1 corresponds to  $p = 3.8 < p_c$ . The curves 2–9 correspond to  $p \geq p_c$ :  $p = p_c$  (curve 2),  $p = 4$  (curve 3),  $p = 4.135$  (curve 4),  $p = 4.3647$  (curve 5),  $p = 4.5824$  (curve 6),  $p = 4.8$  (curve 7),  $p = 5$  (curve 8),  $p = 6$  (curve 9).

The curves plotted in Figure 6 present the isotherms — the dependence of the pressure on  $\bar{y}$ , which is equivalent to the dependence on the density  $\bar{n}$ , see (2.26), calculated from (2.28) for a number of fixed values of  $p$ .

## 4. CONCLUDING REMARKS

In this work, we proved the existence of multiple thermodynamic phases at the same values of the extensive model parameters – temperature and chemical potential. In contrast to the approach of [1], we deal directly with thermodynamic phases in the grand canonical setting. To the best of our knowledge, this is the first result of this kind.

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